The puzzle that led to Noether’s theorems

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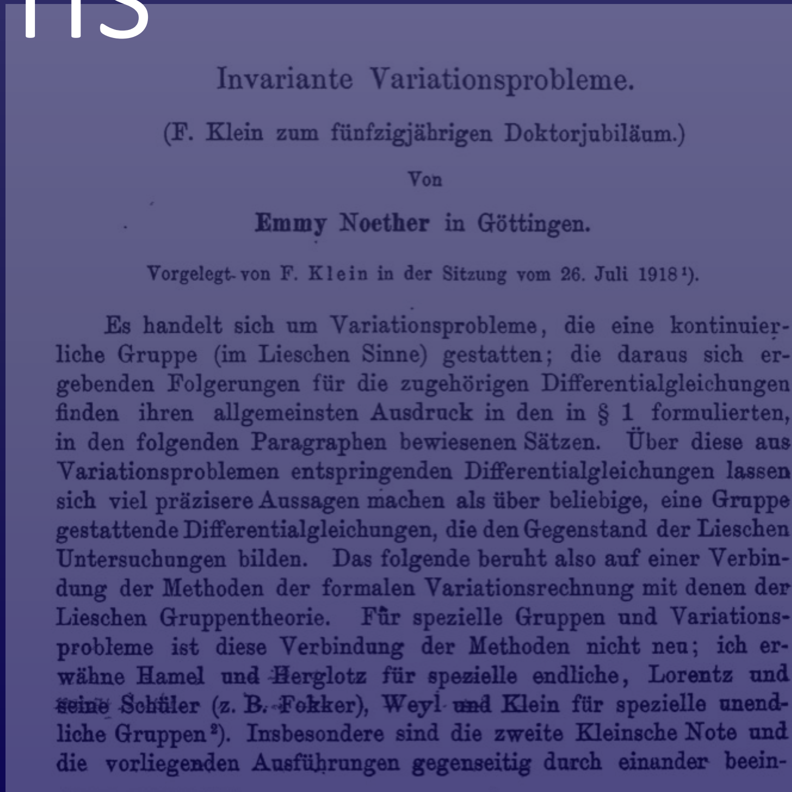

Invariante Variationsprobleme.

(F. Klein zum fünfzigjährigen Doktorjubiläum.)

Von

Emmy Noether in Göttingen.

Vorgelegt von F. Klein in der Sitzung vom 26. Juli 1918[1]).

Es handelt sich um Variationsprobleme, die eine kontinuierliche Gruppe (im Lieschen Sinne) gestatten; die daraus sich ergebenden Folgerungen für die zugehörigen Differentialgleichungen finden ihren allgemeinsten Ausdruck in den in § 1 formulierten, in den folgenden Paragraphen bewiesenen Sätzen. Über diese aus Variationsproblemen entspringenden Differentialgleichungen lassen sich viel präzisere Aussagen machen als über beliebige, eine Gruppe gestattende Differentialgleichungen, die den Gegenstand der Lieschen Untersuchungen bilden. Das folgende beruht also auf einer Verbindung der Methoden der formalen Variationsrechnung mit denen der Lieschen Gruppentheorie. Für spezielle Gruppen und Variationsprobleme ist diese Verbindung der Methoden nicht neu; ich erwähne Hamel und Herglotz für spezielle endliche, Lorentz und seine Schüler (z. B. Fokker), Weyl und Klein für spezielle unendliche Gruppen[2]). Insbesondere sind die zweite Kleinsche Note und die vorliegenden Ausführungen gegenseitig durch einander beein-

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Emmy Noether, 'Invariant Variation Problems', 1918

The paper contains two theorems

The first theorem is the one she is most famous for among physicists

"For every symmetry, a conservation law"

The second theorem is also used by physicists e.g. general relativity, the Bianchi identities

Why did Emmy Noether derive these theorems?
What puzzle was she trying to solve?

When Emmy Noether formulated her two theorems, what puzzle was she trying to solve?

Answer: a puzzle about the status of energy conservation in Einstein's General Theory of Relativity

More specifically: she offered a proof of a claim made by Hilbert (1917) that energy conservation has a different status in Einstein's theory, and that this is characteristic of "generally covariant" theories

Important: Noether's proof needs *both* of her theorems.

Hilbert, 1917, response to Klein

"With your considerations on the energy theorem I am in full factual agreement: with Emmy Noether, whose help I called upon for clarification of questions pertaining to the analytical treatment of my energy theorem more than a year ago, I found accordingly that the energy components set up by me, just as those of Einstein, can be formally transformed by means of the Lagrangian differential equations . . . of my first contribution, into expressions whose divergence identically , that is without reference to the Lagrangian equations [. . .] vanishes.

Since on the other hand the energy equations of classical mechanics, of the theory of elasticity, and of electrodynamics, are fulfilled only as a consequence of the Lagrangian differential equations of these problems, then it is justified if you accordingly do not recognise in my energy equations the analogues of those of your theory.

Certainly I maintain that for general relativity, that is, in the case of general invariance of the Hamiltonian function, [such] energy equations . . . in general do not exist . . . I might designate this circumstance as a characteristic trait of the general theory of relativity. For my assertion, mathematical proof should be adduced."

Noether, ‘Invariant Variation Problems’, 1918

Concluding section of her paper:

“From the foregoing, finally, we also obtain the proof of a Hilbertian assertion about the connection of the lack of proper energy theorems with “general relativity” (Klein’s first note, Göttinger Nachr. 1917, Reply, 1st para), and even in a generalized group theoretic version.”

What’s the issue with “energy theorems” in connection with “general relativity”?

Summer 1915: Einstein visits Göttingen

- Einstein has been working towards a theory of gravity…
  - 1905, publication of special relativity
  - since 1912, series of papers towards a theory of gravity
  - December 1915, publication of the Einstein Field Equations of general relativity

… but he has not yet arrived at general relativity

- Einstein gives six two-hour lectures on gravitation theory (was Noether in the audience?), and meets Hilbert

- “general covariance” and “energy conservation” play a prominent role in his thinking
  - the significance of “general covariance” for Einstein
  - the role of energy conservation in relation to general covariance

“I [i.e. Einstein] was in Göttingen for a week, where I met him [i.e. Hilbert] and became quite fond of him. I held six two-hour lectures there on gravitation theory, which is now clarified very much already, and had the pleasure of convincing the mathematicians there thoroughly.” (Letter to Zangger, Berlin, 7 July 1915)

“In Göttingen I had the great pleasure of seeing that everything was understood down to the details.” (Einstein to Arnold Sommerfeld, Sellin, 15 July 1915)

Summer 1915: Einstein visits Göttingen

Three things Hilbert seems to have taken away:

- search for generally covariant field equations
- need some sort of restriction
- use energy conservation to get that restriction

Hilbert adopts general covariance as an axiom and investigates it consequences for theories of gravitation and electromagnetism

He sees that any generally covariant field equations for electromagnetism and gravity will have “dependency relations” among them

He thinks all will be well physically because of energy conservation

December 1915: Einstein publishes the Einstein Field Equations of his General Theory of Relativity

- generally covariant
- energy conservation no longer used to restrict the covariance of the field equations

This leaves Hilbert with two puzzles

- what about the causality problem (dependency relations)?
- what happens to energy conservation?

Hilbert’s assertion:

Conservation of energy in generally covariant theories has a “different status” than in theories that are not generally covariant
Reason: the dependency relations that arise among the field equations due to general covariance

Hilbert, 1917, response to Klein

"With your considerations on the energy theorem I am in full factual agreement: with Emmy Noether, whose help I called upon for clarification of questions pertaining to the analytical treatment of my energy theorem more than a year ago, I found accordingly that the energy components set up by me, just as those of Einstein, can be formally transformed by means of the Lagrangian differential equations . . . of my first contribution, into expressions whose divergence identically , that is without reference to the Lagrangian equations [. . .] vanishes.

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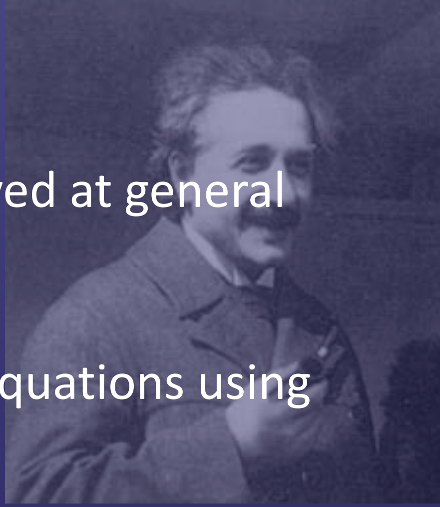
General covariance and energy conservation



Summer 1915: Einstein has been working towards a theory of gravity, but he has not yet arrived at general relativity

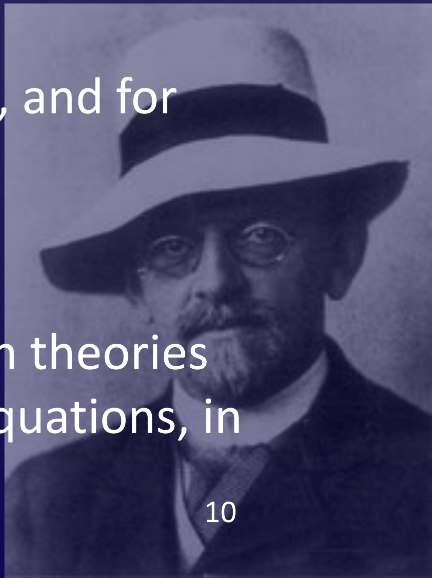
- general covariance and energy conservation play a prominent role in his thinking
- thinks that no generally covariant theory is to be had, restricts the covariance of his field equations using energy conservation

Hilbert seized on general covariance, and put it at the heart of his own investigations into field equations for physics

- sees that any generally covariant field equations for electromagnetism and gravity will have "dependency relations" among them
- thinks all will be well physically because of energy conservation

December 1915: Einstein publishes the Einstein Field Equations, which are generally covariant, and for which energy conservation no longer plays any restricting role

Hilbert revisits his own conclusions concerning the consequences of general covariance
Asserts that energy conservation in generally covariant theories has a "different status" than in theories that are not generally covariant, due to the dependency relations that arise among the field equations, in turn due to general covariance



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Noether’s first theorem: can be used to connect invariance properties of e.g. action functionals to conservation laws

Noether’s second theorem: can be used to show dependency relations among the field equations

Using them both together:

“From the foregoing, finally, we also obtain the proof of a Hilbertian assertion about the connection of the lack of proper energy theorems with “general relativity”…, and even in a generalized group theoretic version.”

Noether's theorems

I. If the integral I is invariant with respect to a $\mathfrak{G}_\rho$, then ρ linearly independent combinations of the Lagrange expressions become divergences — and from this, conversely, invariance of I with respect to a $\mathfrak{G}_\rho$ will follow. The theorem holds good even in the limiting case of infinitely many parameters.

$$E^i \xi_i^k = \partial_\mu j^{k\mu}_{(Noether)}$$

II. If the integral I is invariant with respect to a $\mathfrak{G}_{\infty\rho}$ in which the arbitrary functions occur up to the σ-th derivative, then there subsist ρ identity relationships between the Lagrange expressions and their derivatives up to the σ-th order. In this case also, the converse holds.[6]

$$E^i a_i^k \doteq \partial_\nu \left(b_i^{k\nu} E^i \right)$$

where

- k is one of the ρ parameters
- E^i is the Euler derivative for the field i
- the form that ξ_i^k, a_i^k, and $b_i^{k\nu}$ take depend on the form of the invariance transformation

Noether’s theorems

Mathematical theorems formulated using
- variational techniques
- group theory, Lie groups

Both theorems begin from the same variational problem, applied to some *S* (an integral of some function of independent variables, dependent variables, and their derivatives):

If the first order functional variation in *S* vanishes for an arbitrary region of integration, what general conditions must hold?

Theorem I considers the special case where *S* is invariant under a continuous group of transformations depending on *constant* parameters.

- finite groups, “global” transformations
- the variations we are consider are infinitesimal *global* transformations that leave *S* invariant
- (“global” symmetries are the ones we are familiar with from “classical” theories)

Theorem II considers the special case where *S* is invariant under a continuous group of transformations depending on *arbitrary functions* of the independent variables.

- infinite groups, “local” transformations
- the variations we consider are infinitesimal *local* transformations that leave *S* invariant
- (the “general covariance” of the Einstein Field Equations is an example of a “local” symmetry)

Noether's theorems

Theorem I considers the special case where *S* is invariant under a continuous group of transformations depending on *constant* parameters.

For every constant parameter, there is an expression (in terms of the variables and their derivatives) whose divergence vanishes when the Euler-Lagrange equations are satisfied

$$E^i \xi_i^k = \partial_\mu j^{k\mu}_{(Noether)}$$

When E^i=0, we arrive at a continuity equation

Theorem II considers the special case where *S* is invariant under a continuous group of transformations depending on *arbitrary functions* of the independent variables.

For every arbitrary function, there is a *dependency* among the variables and their derivatives that takes a specific form…

$$E^i a_i^k \doteq \partial_\nu \left(b_i^{k\nu} E^i \right)$$

Hamilton's Principle and Noether's Problem compared

Hamilton's Principle:

- end points fixed
- arbitrary variations
- require the variation in the action to vanish
- upshot: Euler-Lagrange equations

$$E^i = \frac{\partial L}{\partial \phi_i} - \partial_\nu \left(\frac{\partial L}{\partial(\partial_\nu \phi_i)} \right) = 0$$

Noether's Problem:

- end points not fixed
- specific variations…
- … chosen to be those for which the variation in the action vanishes
- upshot: Noether's theorems

$$E^i \xi_i^k = \partial_\mu j^{k\mu}_{(Noether)}$$

$$E^i a_i^k \doteq \partial_\nu \left(b_i^{k\nu} E^i \right)$$

Noether's theorems

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Noether's theorems: the proof of Hilbert's assertion

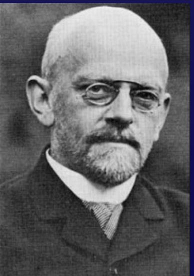
When our theory is invariant under global transformations only, Noether's first theorem applies, and

$$E^i \xi_i^k = \partial_\mu j^{k\mu}_{(Noether)}$$

yields a "proper" conservation law.

Specifically:

$$\partial_\mu j^{k\mu}_{(Noether)} = 0$$



follows as a consequence of $E^i=0$

When our theory is invariant under local transformations, Noether's second theorem applies, and there are dependencies among the Euler derivatives. When the global group of transformations is a subgroup of the local group,

$$\partial_\mu j^{k\mu}_{(Noether)} = 0$$

follows as a consequence of these dependencies, and is therefore "improper".

In general relativity, for which the field equations are generally covariant, energy conservation becomes "improper".

But what is the physical significance of this mathematical result?

Hilbert and Klein think that the status of energy conservation is different in generally covariant theories: the energy theorems become “mathematical identities” rather than being of physical significance.

Einstein disagrees. He writes to Klein (13 March 1918): “I do not find your remark about my formulation of the conservation laws appropriate.”

How do Noether’s theorems help us think about this?

From Noether's first and second theorems we know:

In non-generally covariant theories:

the divergence of the energy expression vanishes only as consequence of the laws, as Hilbert said.

In generally covariant theories:

- general covariance leads to a lack of independence among the Euler derivatives
- these dependency relations lead to the energy expression having a particular form
- the divergence of the energy expression vanishes in virtue of this form, independently of the field equations

so in this sense (of not depending on the field equations, but rather on their mathematical form) Hilbert and Klein are right

But:

the dependency relations of the second theorem are not always physically empty: they express dependencies among the fields (e.g. between the matter fields and the metric in general relativity) and so need not lack physical significance. In this sense, Einstein was right.

Einstein on Noether’s theorems

Einstein wrote to Hilbert: “Yesterday I received from Miss Noether a very interesting paper on invariants. I’m impressed that such things can be understood in such a general way. The old guard at Göttingen should take some lessons from Miss Noether! She seems to know her stuff.”

When Emmy Noether formulated her two theorems, what puzzle was she trying to solve?

Answer: a puzzle about the status of energy conservation in Einstein's General Theory of Relativity

More specifically: she offered a proof of a claim made by Hilbert (1917) that energy conservation has a different status in Einstein's theory, and that this is characteristic of "generally covariant" theories

Important: Noether's proof needs *both* of her theorems.

This talk draws on my D.Phil. completed under the direction of Harvey Brown; joint research on Noether's theorems with Harvey Brown; and K. Brading (2005) "A note on General Relativity, Energy Conservation, and Noether's Theorems", in *The Universe of General Relativity*, Einstein Studies 11, ed. A. J. Kox and J. Eisenstaedt, 125-135.